Double-Angle and Half-Angle Identities

Double-Angle Identities

Using the sum identities from the last section, we can derive more formulas called the double-angle identities:

$$\sin(2x) = \sin(x+x) = \sin x\cos x + \cos x\sin x = 2\sin x\cos x$$

$$\cos(2x) = \cos(x+x) = \cos x\cos x - \sin x\sin x = \cos^2 x - \sin^2 x$$

$$\cos(2x) = \cos^2 x - \sin^2 x = \left(1-\sin^2 x\right) - \sin^2 x = 1 - 2\sin^2 x$$

$$\cos(2x) = \cos^2 x - \sin^2 x = \cos^2 x - \left(1-\cos^2 x\right) = 2\cos^2 x - 1$$

$$\tan(2x) = \tan(x+x) = \frac{\tan x + \tan x}{1 - \tan x\tan x} = \frac{2\tan x}{1-\tan^2 x}$$

Example: Find $\sin\frac{2\pi}{3}$, $\cos\frac{2\pi}{3}$, and $\tan\frac{2\pi}{3}$ using double-angle identities.

Example: Use the double angle identities to verify that $\cos(3x) = \cos^3 x - 3\cos x\sin^2 x$ is an identity.

Trigonometric Values of Double Angles

Now that we have established the double angle identities, we put them to good use in determining trigonometric values of double angles.

Example 4.3.1.

1. Suppose $P(-3,4)$ lies on the terminal side of θ when θ is plotted in standard position. Find $\cos(2\theta)$ and $\sin(2\theta)$. Determine the quadrant in which the terminal side of the angle 2θ lies when it is plotted in standard position.

Solution.

1. Using $x^2+y^2=r^2$, from **Theorem 2.6** in **Section 2.5**, with $x=-3$ and $y=4$, we find $r=\sqrt{x^2+y^2}=5$. Hence, $\cos(\theta)=\frac{x}{r}=-\frac{3}{5}$ and $\sin(\theta)=\frac{y}{r}=\frac{4}{5}$. It follows that

$$\begin{aligned}\cos(2\theta)&=\cos^2(\theta)-\sin^2(\theta) && \textbf{from double angle identity}\\ &=\left(-\frac{3}{5}\right)^2-\left(\frac{4}{5}\right)^2\\ &=-\frac{7}{25}\end{aligned}$$

and

$$\begin{aligned}\sin(2\theta)&=2\sin(\theta)\cos(\theta) && \textbf{from double angle identity}\\ &=2\left(\frac{4}{5}\right)\left(-\frac{3}{5}\right)\\ &=-\frac{24}{25}.\end{aligned}$$

2. If $\sin(\theta)=x$ for $-\frac{\pi}{2}\le\theta\le\frac{\pi}{2}$, find an expression for $\sin(2\theta)$ in terms of x.

Solution

2. If your first reaction to $\sin(\theta)=x$ is that x should be the cosine of θ, then you have indeed learned something. However, context is everything. Here, x is just a variable. It does not necessarily represent the x-coordinate of a point on the Unit Circle. Here, x represents the quantity $\sin(\theta)$, and what we wish to know is how to express $\sin(2\theta)$ in terms of x. We will see more of this kind of thing in Chapter 5 and, as usual, this is something we need for calculus.

   We start with the double angle identity for sine:

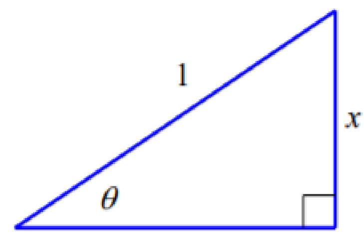
$$\begin{aligned}\sin(2\theta)&=2\sin(\theta)\cos(\theta)\\ &=2x\cos(\theta) && \textbf{from the problem statement that } \sin(\theta)=x\end{aligned}$$

   We need to write $\cos(\theta)$ in terms of x to finish the problem. There are two different methods that come readily to mind, both of which are good to know. The first is purely algebraic, using the Pythagorean identity:

$$\begin{aligned}\cos^2(\theta)+\sin^2(\theta)&=1 && \textbf{Pythagorean identity}\\ \cos^2(\theta)+x^2&=1 && \textbf{from the problem statement}\\ \cos(\theta)&=\pm\sqrt{1-x^2}\\ \cos(\theta)&=\sqrt{1-x^2} && \cos(\theta)\ge 0 \textbf{ since } -\frac{\pi}{2}\le\theta\le\frac{\pi}{2}\end{aligned}$$

The second method, preferred by many, provides a visual approach for determining $\cos(\theta)$. We sketch a right triangle with acute angle θ and, noting that

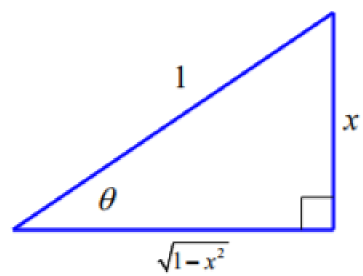
$$\begin{aligned}\sin(\theta) &= x \\ &= \frac{x}{1} = \frac{\text{opposite}}{\text{hypotenuse}},\end{aligned}$$



we label the hypotenuse with length 1 and the side opposite θ with length x. We then use the Pythagorean Theorem to determine the length of the side adjacent to θ.

$$\begin{aligned}(\text{adjacent length})^2 + x^2 &= 1^2 \\ (\text{adjacent length})^2 &= 1 - x^2 \\ \text{adjacent length} &= \sqrt{1-x^2}\end{aligned}$$

This results in the following triangle.



From the triangle we see that $\cos(\theta) = \dfrac{\sqrt{1-x^2}}{1} = \sqrt{1-x^2}$.

Then, back to solving for $\sin(2\theta)$, we have a final answer is $\sin(2\theta) = 2x\sqrt{1-x^2}$.

Half-Angle Identities

We can use the double-angle identities to derive identities for $\sin(x/2)$, $\cos(x/2)$, and $\tan(x/2)$. We call these the half-angle identities.

To get identities for $\cos(x/2)$ and $\sin(x/2)$, we solve two of the equations for $\cos(2x)$ for $\sin x$ and $\cos x$.

$$\begin{aligned}2\cos^2 x - 1 &= \cos(2x) \\ \cos^2 x &= \frac{1+\cos(2x)}{2} \\ \cos x &= \pm\sqrt{\frac{1+\cos(2x)}{2}}\end{aligned}$$

$$\begin{aligned}1 - 2\sin^2 x &= \cos(2x) \\ -2\sin^2 x &= \cos(2x) - 1 \\ \sin^2 x &= \frac{1-\cos(2x)}{2} \\ \sin x &= \pm\sqrt{\frac{1-\cos(2x)}{2}}\end{aligned}$$

Since these equations work for any value of *x*, they also work if we replace *x* by $(x/2)$:

$$\cos\frac{x}{2} = \pm\sqrt{\frac{1+\cos x}{2}} \qquad \sin\frac{x}{2} = \pm\sqrt{\frac{1-\cos x}{2}}$$

We can then use these formulas to derive formulas for $\tan\frac{x}{2}$:

$$\tan\frac{x}{2} = \pm\sqrt{\frac{1-\cos x}{1+\cos x}} \qquad \tan\frac{x}{2} = \frac{\sin x}{1+\cos x} \qquad \tan\frac{x}{2} = \frac{1-\cos x}{\sin x}$$

Examples: Use the half-angle identities to find the exact values of $\sin\frac{\pi}{8}$, $\cos\frac{\pi}{8}$, and $\tan\frac{\pi}{8}$.

Example: Prove that $\sin^2\left(\frac{x}{2}\right)\cos^2\left(\frac{x}{2}\right) = \frac{\sin^2 x}{4}$ is an identity.

Example: Find $\sin\alpha$, $\cos\alpha$, and $\tan\alpha$ if $\cos(2\alpha) = -1/3$ and $\pi < 2\alpha < 3\pi/2$.

Example: Find $\sin\alpha$, $\cos\alpha$, and $\tan\alpha$ if $\sin(\alpha/2) = 4/5$ and $\pi/4 < \alpha/2 < \pi/2$.